

Springer fiber components in the two columns case for types A and D are normal

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Abstract

We study the singularities of the irreducible components of the Springer fiber over a nilpotent element N with $N^2 = 0$ in a Lie algebra of type A or D (the so-called two columns case). We use Frobenius splitting techniques to prove that these irreducible components are normal, Cohen–Macaulay, and have rational singularities.

1 Introduction

Let \mathbb{K} be an algebraically closed field of arbitrary characteristic not equal to 2. Let N be a nilpotent element in a Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ (type A) or $\mathfrak{g} = \mathfrak{so}_{2n}$ (type D). We consider the Springer fiber \mathcal{F}_N over N . It is the fiber of the famous Springer resolution of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ over N .

This resolution can be constructed as follows. Let \mathcal{F} be the variety of complete flags in \mathbb{K}^n (resp. \mathcal{OF} the variety of complete *isotropic* flags, see Section 3 for the description of the Springer fiber \mathcal{OF}_N in this case). A flag $f = (V_i)_{i \in [0, n]}$, where V_i is of dimension i , is stabilized by $N \in \mathcal{N}$ if $N(V_i) \subset V_{i-1}$ for all $i > 0$. We shall denote this by $N(f) \subset f$. Define the variety

$$\tilde{\mathcal{N}} = \{(f, N) \in \mathcal{F} \times \mathcal{N} \mid N(f) \subset f\}.$$

The projection $\tilde{\mathcal{N}} \rightarrow \mathcal{F}$ is a smooth morphism thus $\tilde{\mathcal{N}}$ is smooth. The natural projection $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$ is proper and is therefore a resolution of singularities for \mathcal{N} . It is called the *Springer resolution*.

The *Springer fibers*, i.e., the fibers of the Springer resolution, are of great interest. They are connected (this can be seen directly or follows from the normality of the nilpotent cone \mathcal{N}), equidimensional, but not irreducible. There is a natural combinatorial framework to describe them: Young diagrams and standard tableaux.

The irreducible components of the Springer fibers are not well understood. For example, it is known that in general the components are singular but there is no general description of the singular components. There are only partial answers in type A . First, it is known in the so-called *hook* and *two lines* cases, all the components are smooth (see [Fun03]). The first case where singular components appear is the *two columns* case. A description of the singular components in the two columns case has been given by L. Fresse in [Fre08] and [Fre09]. In their recent work [FrMe09] L. Fresse and A. Melnikov describe the Young diagrams for which all irreducible components are smooth.

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In this paper, we focus on the the two columns case, that is to say, the case of nilpotent elements N of order 2. The corresponding Young diagram $\lambda = \lambda(N)$ has two columns. We want to understand the type of singularities appearing in a component of the Springer fiber.

Let X be an irreducible component of the Springer fiber \mathcal{F}_N , resp. \mathcal{OF}_N , in type A , resp. D , with N nilpotent such that $N^2 = 0$. In the two columns case, we describe a resolution $\pi: \tilde{X} \rightarrow X$ of the irreducible component X . We use this resolution to prove, for $\text{Char}(\mathbb{K}) > 0$, that X is Frobenius split, and deduce the following result for arbitrary characteristic:

Theorem 1.1. *The irreducible component X is normal.*

We are able to prove more on the resolution π . Recall that a proper birational morphism $f: X \rightarrow Y$ is called a rational resolution if X is smooth and if the equalities $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_*\mathcal{O}_X = R^i f_*\omega_X = 0$ for $i > 0$ are satisfied. We prove the following

Theorem 1.2. *The morphism π is a rational resolution.*

Corollary 1.3. *The irreducible component X is Cohen–Macaulay with dualizing sheaf $\pi_*\omega_{\tilde{X}}$.*

Rational singularities are well defined in characteristic zero. In this case we obtain the following

Corollary 1.4. *If $\text{Char}(\mathbb{K}) = 0$, then X has rational singularities.*

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2 Irreducible components of Springer fibers in type A

2.1 General case

Let $N \in \mathfrak{gl}_n$ be a nilpotent element, and let $(m_i)_{i \in [1, r]}$ be the sizes of Jordan blocks of N . To N we assign a Young diagram $\lambda = \lambda(N)$ of weight n with rows of lengths $(m_i)_{i \in [1, r]}$.

Definition 2.1. A *standard Young tableau* of shape λ is a bijection $\tau: \lambda \rightarrow [1, n]$ such that the numbers assigned to the boxes in each row are decreasing from left to right, and the numbers in each column are decreasing from top to bottom.

Remark 2.2. Usually, one requires that the integers in the boxes of a standard tableau *increase*, not decrease from left to right and from top to bottom. However, using *decreasing* tableaux in our case simplifies the notation, so we decided to follow this (rather unusual) definition.

Remark that the datum of a standard tableau τ is equivalent to the datum of a chain of decreasing Young diagrams $\lambda = \lambda^{(0)} \supset \lambda^{(1)} \supset \lambda^{(2)} \supset \dots \supset \lambda^{(n)} = \emptyset$, where $\lambda^{(i)}$ is the set of the $n - i$ boxes with the largest numbers, that is, $\tau^{-1}(\{i + 1, \dots, n\})$.

Let $f = (V_i) \in \mathcal{F}$ be an N -stable flag. We assign to it a standard tableau of shape $\lambda = \lambda(N)$ in the following way. Consider the quotient spaces $V^{(i)} = V/V_i$. The endomorphism N induces an endomorphism of each of these quotients $N^{(i)}: V^{(i)} \rightarrow V^{(i)}$. Take the Young diagram $\lambda^{(i)}$ corresponding to $N^{(i)}$; it consists of $n - i$ boxes. Clearly, $\lambda^{(i)}$ differs from $\lambda^{(i-1)}$ by one corner box. So we obtain a chain of decreasing Young diagrams, which is equivalent to a standard Young tableau $\tau(f)$.

Let τ be a standard Young tableau of shape $\lambda(N)$. Define

$$X_\tau^0 = \{f \in \mathcal{F}_N \mid \tau(f) = \tau\}.$$

The following theorem is due to Spaltenstein [Spa82].

Theorem 2.3. *For each standard tableau τ , the subset X_τ^0 is a smooth irreducible subvariety of \mathcal{F}_N . Moreover, $\dim X_\tau^0 = \dim \mathcal{F}_N$, so $X_\tau = \overline{X_\tau^0}$ is an irreducible component of \mathcal{F}_N . Any irreducible component of \mathcal{F}_N is obtained in this way.*

2.2 Two columns case

In this paper, we focus on the case of nilpotent elements N such that $N^2 = 0$. This is equivalent to saying that the Young diagram $\lambda(N)$ consists of (at most) two columns. Denote by r the rank of N or, equivalently, the number of boxes in the second column. Let $X = X_\tau$ be the irreducible component of the Springer fiber over N corresponding to a standard tableau τ . Denote the increasing sequence of labels in the second column of the standard tableau τ by $(p_i)_{i \in [1, r]}$. Set $p_0 = 0$ and $p_{r+1} = n + 1$.

According to F.Y.C. Fung [Fun03], the previous Theorem can be reformulated as follows.

Proposition 2.4. *The irreducible component X is the closure of the variety*

$$X^0 = \left\{ (V_i)_{i \in [0, n]} \in \mathcal{N}_N \mid \begin{array}{ll} V_i \subset V_{i-1} + \text{Im } N & \text{for } i \in \{p_1, \dots, p_r\} \\ V_i \not\subset V_{i-1} + \text{Im } N & \text{otherwise} \end{array} \right\}.$$

An easy interpretation of this result is the following

Corollary 2.5. *The irreducible component X is the closure of the variety*

$$X^0 = \left\{ (V_i)_{i \in [0, n]} \in \mathcal{N}_N \mid \dim(\text{Im } N \cap V_i) = k \text{ for all } k \in [0, r] \text{ and all } i \in [p_k, p_{k+1}] \right\}.$$

Proof. We prove this by induction on i . We have $\dim(\text{Im}N \cap V_0) = 0$. The result is implied by the following equivalence: $(V_{i+1} \subset V_i + \text{Im}N) \Leftrightarrow (\dim(\text{Im}N \cap V_{i+1}) = \dim(\text{Im}N \cap V_i) + 1)$. \square

2.3 A birational transformation of the Springer fiber

The above description gives a natural way to construct a resolution of singularities for X . We start with the following simple birational transformation of X . Define the variety \widehat{X} as follows:

$$\widehat{X} = \{((F_k)_{k \in [0,r]}, (V_i)_{i \in [0,n]}) \in \mathcal{F}(\text{Im}N) \times \mathcal{F} \mid F_k \subset V_{p_k} \subset N^{-1}(F_{k-1}), \forall k \in [1, r]\},$$

where $\mathcal{F}(\text{Im}N)$ denotes the variety of complete flags in $\text{Im}N$. The natural projections of the product $\mathcal{F}(\text{Im}N) \times \mathcal{F}$ on its two factors induce two maps $p_X : \widehat{X} \rightarrow \mathcal{F}$ and $q_X : \widehat{X} \rightarrow \mathcal{F}(\text{Im}N)$.

One of the main features of the two columns case that we will use is the following easy observation: $\text{Im}N \subset \text{Ker}N$. In particular, for any flag $(F_k)_{k \in [0,r]} \in \mathcal{F}(\text{Im}N)$, the equalities $F_r = \text{Im}N$ and $N^{-1}(F_0) = \text{Ker}N$ imply the following inclusions:

$$F_0 \subset \cdots \subset F_r \subset N^{-1}(F_0) \subset \cdots \subset N^{-1}(F_r).$$

Fixing subspaces $(F_i)_{i \in [r, n-r]}$ with $\dim(F_i) = i$ such that

$$\text{Im}N \subset F_r \subset \cdots \subset F_{n-r} \subset \text{Ker}N$$

gives for any choice of $(F_k)_{k \in [0,r]} \in \mathcal{F}(\text{Im}N)$ a complete flag

$$F_0 \subset \cdots \subset F_r \subset F_{r+1} \subset \cdots \subset F_{n-r-1} \subset N^{-1}(F_0) \subset \cdots \subset N^{-1}(F_r)$$

in $N^{-1}(F_r) = \mathbb{K}^n$. We denote this complete flag by F_\bullet .

Proposition 2.6. (i) *The map q_X is dominant and is a locally trivial fibration over $\mathcal{F}(\text{Im}N)$. Its fiber over $(F_k)_{k \in [0,r]}$ is isomorphic to the following Schubert variety associated to F_\bullet :*

$$\mathcal{F}_w = \{(V_i)_{i \in [0,n]} \in \mathcal{F} \mid F_k \subset V_{p_k} \subset N^{-1}(F_{k-1}), \forall k \in [1, r]\}.$$

(ii) *The map p_X is birational onto X .*

Proof. (i) The first part is clear from the definition of \widehat{X} .

(ii) Let $(V_i)_{i \in [0,n]}$ be in X^0 . We may define $F_k = \text{Im}N \cap V_{p_k}$ for $k \in [0, r]$. We have $\dim F_k = k$. Since $(V_i)_{i \in [0,n]}$ is in the Springer fiber, we also have the inclusion $N(V_{p_k}) \subset V_{p_{k-1}}$. But $N(V_{p_k}) \subset \text{Im}N$, thus

$$N(V_{p_k}) \subset \text{Im}N \cap V_{p_{k-1}}.$$

Since $(V_i)_{i \in [0,n]}$ is in X^0 , we have $\text{Im}N \cap V_{p_{k-1}} = \text{Im}N \cap V_{p_{k-1}}$. Therefore we have the inclusion:

$$V_{p_k} \subset N^{-1}(\text{Im}N \cap V_{p_{k-1}}) = N^{-1}(F_{k-1}).$$

In particular X^0 is contained in the image of p_X .

Conversely, let $(F_k)_{k \in [0,r]} \in \mathcal{F}(\text{Im}N)$ and $(V_i)_{i \in [0,n]}$ in the Schubert variety \mathcal{F}_w associated to F_\bullet . It is easy to check that for $(V_i)_{i \in [0,n]}$ general in the Schubert variety, we have $\text{Im}N \cap V_i = F_k$ for $i \in [p_k, p_{k+1})$. Furthermore, for $i \in [p_k, p_{k+1})$ we have the inclusions

$$N(V_{i+1}) \subset N(V_{p_{k+1}}) \subset \text{Im}N \cap V_{p_k} = F_k \subset V_{p_k} \subset V_i,$$

therefore $(V_i)_{i \in [0,n]}$ is in X^0 . \square

2.4 A Schubert variety containing X

Let us consider the following subvariety of $\mathcal{F}(\text{Im}N) \times \mathcal{F}$ containing \widehat{X} :

$$\widehat{Y} = \{((F_k)_{k \in [0,r]}, (V_i)_{i \in [0,n]}) \in \mathcal{F}(\text{Im}N) \times \mathcal{F} \mid F_k \subset V_{p_k}, \forall k \in [1, r]\}.$$

As for \widehat{X} , the natural projections of the product $\mathcal{F}(\text{Im}N) \times \mathcal{F}$ on its two factors induce two maps $p_Y : \widehat{Y} \rightarrow \mathcal{F}$ and $q_Y : \widehat{Y} \rightarrow \mathcal{F}(\text{Im}N)$.

Proposition 2.7. (i) *The map q_Y is dominant and is a locally trivial fibration with fiber over $(F_k)_{k \in [0,r]}$ isomorphic to the following Schubert variety associated to F_\bullet :*

$$\mathcal{F}_v = \{(V_i)_{i \in [0,n]} \in \mathcal{F} \mid F_k \subset V_{p_k}, \forall k \in [1, r]\}.$$

(ii) *The map p_Y is birational onto the Schubert variety*

$$Y = \{(V_i)_{i \in [0,n]} \in \mathcal{F} \mid \dim(\text{Im}N \cap V_{p_k}) \geq k, \forall k \in [1, r]\}.$$

Proof. (i) The first part is clear from the definition of \widehat{Y} .

(ii) The image of p_Y is contained in Y . Conversely, let $(V_i)_{i \in [0,n]}$ be general in Y . We then have $\dim(\text{Im}N \cap V_{p_k}) = k$ and we may define $F_k = \text{Im}N \cap V_{p_k}$ for $k \in [0, r]$. We have $\dim F_k = k$ and $((F_k)_{k \in [0,r]}, (V_i)_{i \in [0,n]})$ is in the fiber of p_Y over $(V_i)_{i \in [0,n]}$. \square

3 Irreducible components of Springer fibers in type D

3.1 Preliminaries on orthogonal groups and Springer fibers

Let V be a $2n$ -dimensional vector space. Consider the group $\text{SO}(V)$ of unimodular linear operators preserving a symmetric nondegenerate bilinear form ω . Let B be a Borel subgroup in $\text{SO}(V)$. The flag variety $\text{SO}(V)/B$ is the variety OF of *orthogonal flags* defined by

$$\text{OF} = \{V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_{n+1} \subset \cdots \subset V_{2n} \mid V_{2n-i} = V_i^\perp \text{ and } \dim V_i = i \text{ for } i \leq n-1\}.$$

We will consider elements in OF as n -tuples of nested isotropic vector spaces $((V_i)_{i \in [0, n-1]})$. We recover the usual notion of orthogonal flags because there are exactly two maximal isotropic subspaces between V_{n-1} and its orthogonal $V_{n+1} = (V_{n-1})^\perp$.

Let $N \in \mathfrak{gl}(V)$ be a nilpotent element. N is said to be *orthogonalizable* if there exists a symmetric nondegenerate bilinear form ω on V such that N is ω -invariant; that is,

$$\omega(Nv, w) + \omega(v, Nw) = 0.$$

This means that $N \in \mathfrak{so}(V)$, where $\mathfrak{so}(V)$ is the set of elements of $\mathfrak{gl}(V)$ leaving ω invariant. The following easy consequence of the Jacobson–Morozov Theorem can be found, for instance, in [ViGoOn90, Chap. 6, 2.3].

Proposition 3.1. *A nilpotent element N is orthogonalizable if in the corresponding partition $Y(N)$ each even term occurs with even multiplicity (such partitions will be called admissible).*

Definition 3.2. Given a nilpotent element $N \in \mathfrak{so}(V)$, we define a *Springer fiber of type D* in the usual way: namely, as the set of all orthogonal flags stabilized by N :

$$\text{OF}_N = \{((V_i)_{i \in [0, n-1]}) \in \text{OF} \mid N(V_i) \subset V_{i-1} \text{ for } i \in [0, n-1] \text{ and } N(V_{n-1}^\perp) \subset V_{n-1}\}.$$

Remark that the orthogonalizability condition on N implies that $N(V_i^\perp) \subset V_{i+1}^\perp$. A description of irreducible components of Springer fibers in types B , C , and D was given in M. van Leeuwen's Ph.D. thesis [vLe89]. We briefly recall this description here for the type D .

Definition 3.3. Let λ be a Young diagram with $2n$ boxes. A map τ from the boxes of λ to $[1, n]$ is called a *standard domino tableau*, if the following conditions hold:

- (i) For each i , the pre-image $\tau^{-1}(i)$ consists exactly of adjacent two boxes (adjacent either by horizontal or by vertical);
- (ii) For each i , the set of boxes $\lambda^{(i)}(N) := \tau^{-1}([i+1, n])$ corresponding to the numbers greater than i forms a Young diagram.

Moreover, a standard domino tableau is said to be *admissible*, if all the diagrams $\lambda^{(i)}(N)$ are admissible (in the sense of Prop. 3.1).

We will think of the pair of boxes $\tau^{-1}(i)$ as of a domino tile indexed by the number i . Each of these tiles can be either horizontal or vertical.

Example 3.4. Let $\lambda = (3, 3)$. Then there are three standard domino tableaux of shape λ (see below), but only the first two of them are admissible. Indeed, for the third diagram $\tau^{-1}(3)$ corresponds to the Young diagram with one row of length 2, which is not admissible.

3	2	1	3	2	2	3	3	1
3	2	1	3	1	1	2	2	1

Definition 3.5. Let τ be an admissible standard domino tableau of shape $\lambda(N)$. We assign to it a subset X_τ of the Springer fiber OF_N obtained as the closure of the set X_τ^0 of flags $(V_i)_{i \in [0, n-1]}$ in OF_N such that $N|_{V_i^\perp/V_i}$ corresponds to the partition $\tau^{-1}([i+1, n])$ for each $i < n$.

By definition of OF_N , N is well defined on $V_i^\perp/V_i = V_{2n+1-i}/V_i$, so this makes sense.

The following theorem is due to M. van Leeuwen [vLe89, Sec. 3.2].

Theorem 3.6. X_τ is an irreducible component of OF_N ; all its irreducible components are obtained in this way. In particular, there is a bijection between the admissible standard domino tableaux of shape $\lambda(N)$ and the irreducible components of OF_N .

3.2 Description of components in the two columns case

Throughout this subsection we fix a nilpotent element $N \in \mathfrak{so}(V)$ such that $N^2 = 0$, and an admissible standard domino tableau τ of shape $\lambda(N)$. We begin with the following combinatorial observation.

Proposition 3.7. The Young diagram $\lambda(N)$ has at most two columns. Then each admissible standard domino tableau of shape $\lambda(N)$ contains only vertical tiles.

As in the type A case (see Corollary 2.5), the description of irreducible components can be reformulated as follows. Let $\text{rk}(N) = 2r$, and let $(p_i)_{i \in [1, r]}$ be the numbers of domino tiles forming the second column of the diagram $Y(N)$ and set $p_0 = 0$, $p_{r+1} = 2n + 1$.

Proposition 3.8. The variety X_τ is the closure of the variety $X_\tau^0 \subset \text{OF}_N$ given by the following equivalent conditions:

- $\dim(N(V_{i-1}^\perp) \cap V_i) = k$ for $i \in [p_k, p_{k+1})$ for all $k \in [0, r]$.

- $\dim(\text{Im}N \cap V_i) = k$ for $i \in [p_k, p_{k+1})$ for all $k \in [0, r]$.

Proof. The first condition is a reformulation of Theorem 3.6 due to M. van Leeuwen [vLe89, Sec. 3.2]. The second condition is seen to be equivalent to the first one by induction on i . For $i = 1$, we have $\text{Im}N \cap V_1 = N(V_0^\perp) \cap V_1$. We then proceed by induction replacing N by its restriction to V_1^\perp/V_1 . \square

3.3 Birational transformation of the Springer fiber

In this subsection we construct a birational transformation of a given irreducible component $X = X_\tau$, analogous to the one described in Section 2.3

First, let us endow the subspace $\text{Im}N$ with a bilinear form α as follows. For $u, v \in \text{Im}N$,

$$\alpha(u, v) = \omega(u, v'), \text{ where } v' \in N^{-1}(v).$$

Proposition 3.9. α is a skew-symmetric nondegenerate form on $\text{Im}N$.

Proof. We readily see that α is well-defined. To show that it is skew-symmetric, take two vectors $u, v \in \text{Im}N$ along with their preimages $u' \in N^{-1}(u), v' \in N^{-1}(v)$. Then

$$\alpha(u, v) = \omega(N(u'), v') = -\omega(u', N(v')) = -\omega(N(v'), u') = -\alpha(v, u).$$

The non-degeneracy of α is also obvious. \square

Remark 3.10. This is a particular case of the construction of a family of nondegenerate bilinear forms on $(\text{Ker}N \cap \text{Im}N^i)/(\text{Ker}N \cap \text{Im}N^{i+1})$, which works for arbitrary nilpotent $N \in \mathfrak{so}(V)$. See [vLe89, Section 2.3] for details.

We shall denote by \angle the orthogonality relation for the form α . We consider the *symplectic flag variety* $\text{Sp}\mathcal{F}(\text{Im}N)$, defined as follows:

$$\text{Sp}\mathcal{F}(\text{Im}V) = \{(0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{2r} = \text{Im}N) \mid F_{2r-k+1} = F_k^\angle\}.$$

Now we are ready to define a birational transformation of X . We define it as follows:

$$\widehat{X} = \{((F_k)_{k \in [0, 2r]}, (V_i)_{i \in [0, n-1]}) \in \text{Sp}\mathcal{F}(\text{Im}N) \times \mathcal{OF} \mid F_k \subset V_{p_k} \subset N^{-1}(F_{k-1}) \quad \forall k \in [0, r]\}. \quad (1)$$

Let us remark that for a flag $(F_k)_{k \in [0, 2r]} \in \text{Sp}\mathcal{F}(\text{Im}N)$, we may consider the partial flag

$$F_0 \subset \cdots \subset F_{2r} = \text{Im}N \subset \text{Ker}N = N^{-1}(F_0) \subset N^{-1}(F_1) \subset \cdots \subset N^{-1}(F_{2r})$$

and since $(F_k)_{k \in [0, 2r]}$ is isotropic for the form α , we have $F_k^\perp = N^{-1}(F_k^\angle) = N^{-1}(F_{2r-k})$. Therefore the above partial flag is isotropic for the quadratic form ω and we may complete it to an isotropic complete flag. As for the type A , denote the two natural projections by $p: \widehat{X} \rightarrow \text{Sp}\mathcal{F}(\text{Im}N)$ and $q: \widehat{X} \rightarrow \mathcal{OF}$ respectively.

Proposition 3.11. (i) The map q is dominant and a locally trivial fibration with fiber isomorphic to the following Schubert variety:

$$\mathcal{OF}_w = \{(V_i)_{i \in [0, 2n]} \in \mathcal{OF} \mid F_k \subset V_{p_k} \subset N^{-1}(F_{k-1}) \quad \forall k \in [1, r]\};$$

(ii) The map p is birational onto X .

Proof. (i) This is clear from the definition of \widehat{X} .

(ii) Let $(V_i)_{i \in [0, 2n]}$ be in X^0 . We use the first characterization of X^0 given by Proposition 3.8. Set $F_k = V_{p_k} \cap N(V_{p_{k-1}}^\perp)$ for $k \in [0, r]$. We have $\dim F_k = k$. Then, $F_r = V_{p_r} \cap N(V_{p_{r-1}}^\perp)$ is α -isotropic. It suffices to show that $\omega(V_{p_r}, N(V_{p_{r-1}}^\perp)) = 0$. But this is so, since $N(V_{p_{r-1}}^\perp) \subset V_{p_r}$.

Let us check that $V_{p_k} \subset N^{-1}(F_{k-1})$. Indeed, we have the inclusions $N(V_{p_k}) \subset V_{p_{k-1}}$ and $V_{p_k} \subset V_{p_{k-1}}^\perp$, so

$$N(V_{p_k}) \subset V_{p_{k-1}} \cap N(V_{p_{k-1}}^\perp) = V_{p_{k-1}} \cap N(V_{p_{k-1}}^\perp) = F_{k-1}.$$

This gives us the desired assertion. So, we have proved that $X^0 \subset \text{Imp}$.

To complete the proof, we need to show that for a given $(F_k)_{k \in [0, 2r]} \in \text{Sp}\mathcal{F}(\text{Im}N)$, a general $(V_i)_{i \in [0, n-1]} \in \text{O}\mathcal{F}$ satisfying (1) is in X^0 . For this we use the second description from Proposition 3.8. For a general element, we have the required dimension equalities and for $i \in [p_k, p_{k+1}]$ we see that

$$N(V_i) \subset N(V_{p_{k+1}}) \subset F_k \subset V_{p_k} \subset V_i.$$

The proposition is proved. \square

3.4 A Schubert variety containing X

As in type A , let us consider the following subvariety of $\text{Sp}\mathcal{F}(\text{Im}N) \times \text{O}\mathcal{F}$ containing \widehat{X} :

$$\widehat{Y} = \{((F_k)_{k \in [0, 2r]}, (V_i)_{i \in [0, n-1]}) \in \text{Sp}\mathcal{F}(\text{Im}N) \times \text{O}\mathcal{F} \mid F_k \subset V_{p_k}, \forall k \in [1, r]\}.$$

As for \widehat{X} , the natural projections of the product $\text{Sp}\mathcal{F}(\text{Im}N) \times \text{O}\mathcal{F}$ on its two factors induce two maps $p_Y : \widehat{Y} \rightarrow \text{O}\mathcal{F}$ and $q_Y : \widehat{Y} \rightarrow \text{Sp}\mathcal{F}(\text{Im}N)$.

Proposition 3.12. (i) *The map q_Y is dominant and is a locally trivial fibration with fiber over $(F_k)_{k \in [0, 2r]}$ isomorphic to the following Schubert variety*

$$\text{O}\mathcal{F}_v = \{(V_i)_{i \in [0, n-1]} \in \text{O}\mathcal{F} \mid F_k \subset V_{p_k}, \forall k \in [1, r]\}.$$

(ii) *The map p_Y is birational onto the Schubert variety*

$$Y = \{(V_i)_{i \in [0, n-1]} \in \text{O}\mathcal{F} \mid \dim(\text{Im}N \cap V_{p_k}) \geq k, \forall k \in [1, r]\}.$$

Proof. (i) The first part is clear from the definition of \widehat{Y} .

(ii) The image of p_Y is contained in Y . Conversely, let $(V_i)_{i \in [0, n]}$ be general in Y . Then the equalities in the second condition of Proposition 3.8 are satisfied. By the same proof as in that proposition, the first conditions are also satisfied. We set $F_k = \text{Im}N \cap V_{p_k}$ for $k \in [0, r]$. This space is isotropic for \angle by the same argument as in the proof of Proposition 3.11. The point $((F_k)_{k \in [0, 2r]}, (V_i)_{i \in [0, n-1]})$ with $F_k = F_{2r+1-k}^\angle$ for $k > r$ is in the fiber of p_Y over $(V_i)_{i \in [0, n-1]}$. \square

4 Frobenius splitting

In this section, we assume $\text{Char}(\mathbb{K}) = p > 0$, we shall intensively use the results from the book [BrKu05]. We refer to this book for the notions of Frobenius splitting of a scheme X and B' -canonical splitting of a scheme X with an action of a Borel subgroup B' of a reductive group G' .

We will now deal with type A and type D simultaneously and keep the notation of the previous two sections. In particular X will denote a fixed irreducible component of the Springer fiber with two columns in type A or D . To simplify the notation we will denote by \mathcal{F}_w both the Schubert variety \mathcal{F}_w in type A and the Schubert variety $\text{O}\mathcal{F}_w$ in type D . We denote by \underline{n} the number n in type A and $n - 1$ in type D .

4.1 Bott–Samelson resolutions

We will use Bott–Samelson resolutions of the Schubert varieties \mathcal{F}_w and \mathcal{F}_v to construct resolutions of \hat{X} and \hat{Y} and thus of X and Y . Let us fix some notation and recall some basic facts on Bott–Samelson resolutions (for details we refer to [Dem74] or [BrKu05]).

Recall that the Schubert varieties in \mathcal{F} are indexed by the elements u of the Weyl group W . The inclusion of Schubert varieties induces an order on W called the *Bruhat order*. Any element $u \in W$ can be written as a product $s_{i_1} \cdots s_{i_k}$ where s_{i_k} is the simple reflection with respect to the simple root α_{i_k} (we shall use the notation of N. Bourbaki here [Bou54]). An expression of minimal length is called *reduced*, and its length k is called the *length* of u . Let us also denote, for α a simple root, by $\mathbb{G}(\alpha)$, respectively $\mathcal{F}(\alpha)$, the Grassmannian (classical in type A , orthogonal in type D), respectively the partial flag variety of all subspaces except those in $\mathbb{G}(\alpha)$. Denote by p_α the projection $\mathcal{F} \rightarrow \mathcal{F}(\alpha)$. Its fiber is isomorphic to \mathbb{P}^1 .

Let F_\bullet be a fixed complete flag (classical in type A , orthogonal in type D), and let $\mathbf{u} = (\alpha_{i_1}, \dots, \alpha_{i_k})$ be a sequence of simple roots. We construct a variety $\tilde{\mathcal{F}}_{\mathbf{u}}$ from these data. For this we consider the following elementary construction.

Elementary construction 4.1. Having a simple root α , we first define a variety

$$\mathcal{F}_\alpha = \{((V_i)_{i \in [0, \underline{n}]}, W) \in \mathcal{F} \times \mathbb{G}(\alpha) \mid W \in p_\alpha^{-1}(p_\alpha(V_i))_{i \in [0, \underline{n}]}\}.$$

There are two natural maps φ_α and ψ_α from \mathcal{F}_α to \mathcal{F} defined by $\varphi_\alpha((V_i)_{i \in [0, \underline{n}]}, W) = (V_i)_{i \in [0, \underline{n}]}$ and $\psi_\alpha((V_i)_{i \in [0, \underline{n}]}, W) = (p_\alpha((V_i)_{i \in [0, \underline{n}]}, W))$. Remark that there is a natural section $\sigma_\alpha : \mathcal{F} \rightarrow \mathcal{F}_\alpha$ of φ_α given by $(V_i)_{i \in [0, \underline{n}]} \mapsto ((V_i)_{i \in [0, \underline{n}]}, p_\alpha((V_i)_{i \in [0, \underline{n}]})$.

Let $p_Z : Z \rightarrow \mathcal{F}$ be a morphism. We define the variety Z_α as the fiber product

$$\begin{array}{ccc} Z_\alpha = Z \times_{\mathcal{F}} \mathcal{F}_\alpha & \longrightarrow & Z \\ \downarrow & & \downarrow p_Z \\ \mathcal{F}_\alpha & \xrightarrow{\varphi_\alpha} & \mathcal{F}. \end{array}$$

We denote the projection $Z_\alpha \rightarrow Z$ by f_{Z_α} . The section σ_α induces a section σ_{Z_α} of f_{Z_α} . We define the map $p_{Z_\alpha} : Z_\alpha \rightarrow \mathcal{F}$ as the composition of the projection $Z_\alpha \rightarrow \mathcal{F}_\alpha$ with ψ_α .

The Bott–Samelson variety $\tilde{\mathcal{F}}_{\mathbf{u}}$ is constructed from the sequence of integers \mathbf{u} and the point F_\bullet in \mathcal{F} . Indeed, we set $Z_0 = \{F_\bullet\}$ with the map $p_{Z_0} : Z_0 \rightarrow \mathcal{F}$ given by the inclusion and we define $Z_1 = (Z_0)_{\alpha_{i_1}}$ obtained by the elementary construction from p_{Z_0} and α_{i_1} . We define by induction $Z_{j+1} = (Z_j)_{\alpha_{i_{j+1}}}$ obtained by the elementary construction from p_{Z_j} and $\alpha_{i_{j+1}}$. By definition, the Bott–Samelson variety $\tilde{\mathcal{F}}_{\mathbf{u}}$ is Z_k . The map $f_{Z_j} : Z_j \rightarrow Z_{j-1}$ is a \mathbb{P}^1 -bundle for all j and therefore $\tilde{\mathcal{F}}_{\mathbf{u}}$ is smooth. The sections σ_{Z_j} define divisors $D_j = f_{Z_k}^{-1} \cdots f_{Z_{j+1}}^{-1} \sigma_{Z_j}(Z_{j-1})$. These divisors intersect transversally, and we define

$$D_J = \bigcap_{j \in J} D_j$$

for $J \subset [1, k]$. For such a subset J of $[1, k]$ we can consider the subword $\mathbf{u}_J = (\alpha_{i_j})_{j \notin J}$ and there is a natural isomorphism $\tilde{\mathcal{F}}_{\mathbf{u}_J} \simeq D_J$. We will therefore consider the Bott–Samelson varieties $\tilde{\mathcal{F}}_{\mathbf{u}'}$ for any subword \mathbf{u}' of \mathbf{u} as subvarieties of the Bott–Samelson variety $\tilde{\mathcal{F}}_{\mathbf{u}}$. We shall denote the union of the divisors D_j for $j \in [1, k]$ by $\partial\tilde{\mathcal{F}}_{\mathbf{u}}$.

Recall that by the construction, there is a map $p_{\tilde{\mathcal{F}}_{\mathbf{u}}} : \tilde{\mathcal{F}}_{\mathbf{u}} \rightarrow \mathcal{F}$. If \mathbf{u} is a reduced expression for an element u of the Weyl group W , the natural map $p_{\tilde{\mathcal{F}}_{\mathbf{u}}}$ is birational onto \mathcal{F}_u yielding a resolution of the Schubert variety \mathcal{F}_u .

Remark 4.2. The choice of a Bott–Samelson resolution $\tilde{\mathcal{F}}_{\mathbf{u}}$ for \mathcal{F}_u depends on the choice of a reduced expression for u . Recall from [Dem74] that since \mathcal{F}_w is a Schubert subvariety of \mathcal{F}_v , we may choose a reduced expression $\mathbf{v} = (\alpha_{i_1} \cdots \alpha_{i_k} \cdots \alpha_{i_l})$ for v such that $\mathbf{w} = (\alpha_{i_k} \cdots \alpha_{i_l})$ is a reduced expression for w . In particular in the diagram

$$\begin{array}{ccc} \tilde{\mathcal{F}}_{\mathbf{w}} & \hookrightarrow & \tilde{\mathcal{F}}_{\mathbf{v}} \\ \downarrow & & \downarrow \\ \mathcal{F}_w & \hookrightarrow & \mathcal{F}_v \end{array}$$

the vertical maps are birational and thus simultaneous resolutions of singularities. We choose such a reduced expression \mathbf{v} for v to construct $\tilde{\mathcal{F}}_{\mathbf{v}}$ and thus $\tilde{\mathcal{F}}_{\mathbf{w}}$.

4.2 Resolutions of X and Y

4.2.1 Two groups

In this subsection, we will need to distinguish the type A and D cases.

In type A , set $G = \mathrm{SL}(\mathrm{Im}N)$ and $G' = \mathrm{SL}(\mathbb{K}^n)$. We embed G in G' as follows. Choose a complement E_1 for $\mathrm{Im}N$ in $\mathrm{Ker}N$ and a complement E_2 for $\mathrm{Ker}N$ in \mathbb{K}^n . We consider the subgroup G_0 of G' defined by:

$$G_0 = \left\{ f \in G' \mid \begin{array}{l} f \text{ stabilizes } \mathrm{Im}N, \mathrm{Ker}N, \text{ and } E_i \text{ for } i \in \{1, 2\}, \\ \det(f|_{\mathrm{Im}N}) = \det(f|_{E_2}) = 1, \text{ and } f|_{E_1} = \mathrm{id}_{E_1} \end{array} \right\}.$$

This group is isomorphic to the product $\mathrm{SL}(\mathrm{Im}N) \times \mathrm{SL}(E_2)$. Furthermore, observe that E_2 is identified with $\mathbb{K}^n/\mathrm{Ker}N$ and with $\mathrm{Im}N$ via N . The group G_0 is thus isomorphic to $G \times G$; let us embed G into G_0 diagonally. For any Borel subgroup B of G , we may find a Borel subgroup B' of G' such that $B \subset B'$ in this embedding. We may thus consider the variety $Z_0 = G/B$ as a subvariety of G_0/B_0 and also of $G'/B' = \mathcal{F}$. We thus have a map $p_{Z_0} : Z_0 \rightarrow \mathcal{F}$.

In type D , set $G = \mathrm{Sp}(\mathrm{Im}N)$ (recall that we have a non degenerate skew form on $\mathrm{Im}N$) and $G' = \mathrm{SO}(\mathbb{K}^{2n})$. We embed G in G' as follows. Choose an orthogonal complement E_1 for $\mathrm{Im}N$ in $\mathrm{Ker}N$ and an isotropic subspace E_2 in \mathbb{K}^n mapping bijectively to $\mathbb{K}^{2n}/\mathrm{Ker}N$. We consider the subgroup G_0 of G' defined by:

$$G_0 = \left\{ f \in G' \mid \begin{array}{l} f \text{ stabilises } \mathrm{Im}N, \mathrm{Ker}N, \text{ and } E_i \text{ for } i \in \{1, 2\}, \\ \det(f|_{\mathrm{Im}N}) = \det(f|_{E_2}) = 1, \text{ and } f|_{E_1} = \mathrm{id}_{E_1} \end{array} \right\},$$

remark that here the conditions $f(\mathrm{Ker}N) \subset \mathrm{Ker}N$ and $\det(f|_{E_2}) = 1$ are redundant. This group is isomorphic to $\mathrm{SL}(\mathrm{Im}N)$. The group G embeds into G_0 . For any Borel subgroup B of G , we may find a Borel subgroup B' of G' such that $B \subset B'$ in this embedding. We may thus consider the variety $Z_0 = G/B$ as a subvariety of G_0/B_0 and also of $G'/B' = \mathcal{F}$. We thus have a map $p_{Z_0} : Z_0 \rightarrow \mathcal{F}$.

4.2.2 Resolutions

We again deal with types A and D simultaneously.

Let us take a sequence of simple roots $\mathbf{u} = (\alpha_{i_1}, \dots, \alpha_{i_k})$ and apply the same construction as for the Bott–Samelson variety $\tilde{\mathcal{F}}_{\mathbf{u}}$, but starting with $Z_0 = G/B$. We get a variety $\tilde{X}_{\mathbf{u}}$ together with a morphism $p_{\tilde{X}_{\mathbf{u}}} : \tilde{X}_{\mathbf{u}} \rightarrow \mathcal{F}$. This variety can also be seen as the homogeneous fiber bundle $\tilde{X}_{\mathbf{u}} = G \times^B \tilde{\mathcal{F}}_{\mathbf{u}}$ where the action of B on $\tilde{\mathcal{F}}_{\mathbf{u}}$ is induced by the inclusion $B \subset B'$ and the natural action of B' on $\tilde{\mathcal{F}}_{\mathbf{u}}$. For any subword \mathbf{u}' of \mathbf{u} , the variety $\tilde{X}_{\mathbf{u}'}$ can again be realized as a complete intersection in $\tilde{X}_{\mathbf{u}}$. In particular we have the same description of divisors $G \times^B D_j$ for $j \in [1, k]$ on $\tilde{X}_{\mathbf{u}}$ as on $\tilde{\mathcal{F}}_{\mathbf{u}}$. We shall denote the union of these divisors by $\partial\tilde{X}_{\mathbf{u}}$. Finally, we have a natural map $p_{\tilde{X}_{\mathbf{u}}} : \tilde{X}_{\mathbf{u}} \rightarrow \mathcal{F}$.

Using the reduced expression \mathbf{v} of v defined in Remark 4.2, we obtain a variety $\tilde{Y} = \tilde{X}_{\mathbf{v}}$ and a subvariety $\tilde{X} = \tilde{X}_{\mathbf{w}}$ of \tilde{Y} . We have natural maps $p_{\tilde{Y}} = p_{\tilde{X}_{\mathbf{v}}}$ (resp. $p_{\tilde{X}} = p_{\tilde{X}_{\mathbf{w}}}$) from \tilde{Y} (resp. \tilde{X}) to \mathcal{F} . Since the maps $p_{\tilde{\mathcal{F}}_{\mathbf{v}}}$ and $p_{\tilde{\mathcal{F}}_{\mathbf{w}}}$ are B' -equivariant and thus B -equivariant, we get a diagram

$$\begin{array}{ccc} \tilde{X} = G \times^B \tilde{\mathcal{F}}_{\mathbf{w}} & \hookrightarrow & \tilde{Y} = G \times^B \tilde{\mathcal{F}}_{\mathbf{v}} \\ \downarrow & & \downarrow \\ \hat{X} = G \times^B \mathcal{F}_{\mathbf{w}} & \hookrightarrow & \hat{Y} = G \times^B \mathcal{F}_{\mathbf{v}} \\ & \searrow & \downarrow \\ & & \mathcal{F}. \end{array}$$

where the morphisms $G \times^B \mathcal{F}_{\mathbf{v}} \rightarrow \mathcal{F}$ and $G \times^B \mathcal{F}_{\mathbf{w}} \rightarrow \mathcal{F}$ are given by $(g, x) \mapsto g \cdot x$. The maps $p_{\tilde{Y}}$ and $p_{\tilde{X}}$ are the vertical compositions in the above diagram. We also have the projection maps $q_{\tilde{Y}} : \tilde{Y} \rightarrow G/B$ and $q_{\tilde{X}} : \tilde{X} \rightarrow G/B$.

Proposition 4.3. (i) *The maps $q_{\tilde{Y}}$ and $q_{\tilde{X}}$ are dominant and locally trivial fibrations with fiber over $(F_k)_{k \in [0, r]} \in G/B$ isomorphic to Bott–Samelson varieties $\tilde{\mathcal{F}}_{\mathbf{v}}$ and $\tilde{\mathcal{F}}_{\mathbf{w}}$, respectively.*

(ii) *The maps $p_{\tilde{Y}} : \tilde{Y} \rightarrow \mathcal{F}$ and $p_{\tilde{X}} : \tilde{X} \rightarrow \mathcal{F}$ are birational and dominant onto Y and X respectively. Thus they are resolutions of singularities for Y and X .*

Proof. The first part is clear from the definition of \tilde{Y} and \tilde{X} . The second part follows from the birationality of the Bott–Samelson resolutions $\tilde{\mathcal{F}}_{\mathbf{v}} \rightarrow \mathcal{F}_{\mathbf{v}}$ and $\tilde{\mathcal{F}}_{\mathbf{w}} \rightarrow \mathcal{F}_{\mathbf{w}}$, the smoothness of \tilde{Y} and \tilde{X} , and the first part. \square

Notation 4.4. For a subword \mathbf{u} of \mathbf{v} , we define $X_{\mathbf{u}}$ to be the subvariety of Y obtained as the image of $\tilde{X}_{\mathbf{u}}$ (seen as a subvariety of $\tilde{Y} = \tilde{X}_{\mathbf{v}}$) under the map $p_{\tilde{Y}}$. With this notation $X = X_{\mathbf{w}}$. The map $p_{\tilde{X}} : \tilde{X} \rightarrow X$ is the resolution π in Theorem 1.2.

4.3 Existence of a splitting

We have the following

Theorem 4.5. (i) *There exists a B' -canonical splitting of the Bott–Samelson variety $\tilde{\mathcal{F}}_{\mathbf{v}}$ compatibly splitting all Bott–Samelson subvarieties $\tilde{\mathcal{F}}_{\mathbf{u}}$ of $\tilde{\mathcal{F}}_{\mathbf{v}}$ for each subword \mathbf{u} of \mathbf{v} .*

(ii) *This splitting induces a B -canonical splitting of \tilde{Y} compatibly splitting all the subvarieties $\tilde{X}_{\mathbf{u}}$ for \mathbf{u} a subword of \mathbf{v} .*

(iii) *The latter splitting induces a splitting of Y compatibly splitting all the subvarieties $X_{\mathbf{u}}$, where \mathbf{u} is a subword of \mathbf{v} .*

Proof. (i) This is an application of [BrKu05, Proposition 4.1.17].

(ii) We first observe that the B' -canonical splitting in (i) is a B_0 -canonical splitting, where $B_0 = B' \cap G_0$ and G_0 was defined in section 4.2. For this, use the following result (see [BrKu05, Lemma 4.1.6]): let H be a connected and simply connected semisimple group, let H' be a Borel subgroup in H , and let H'' be a maximal torus in H . Let X be a H' -scheme and let $\phi \in \text{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$, where F is the Frobenius morphism. Let us denote by $e_\alpha^{(n)}$ the divided powers, where α is a root of H . There exists a natural action $e_\alpha^{(n)} * \phi$ of $e_\alpha^{(n)}$ on ϕ (see [BrKu05, Definition 4.1.4]).

Lemma 4.6. *The element ϕ is a H' -canonical splitting if and only if ϕ is H'' -invariant and $e_\alpha^{(n)} * \phi = 0$ for all $n \geq p$ and α a simple root.*

In our situation, we easily check that the divided powers of G_0 are divided powers for G' . In particular the splitting in (i) is a B_0 -canonical splitting and compatibly splits all the Schubert varieties $\mathcal{F}_u(\mathbb{K}^n)$.

To prove that the B_0 -canonical splitting induces a B -canonical splitting, we need to use results of W. van der Kallen [vdK01, Lemma 10]:

Theorem 4.7. *Let G and H be semi-simple and simply connected groups and assume that H is a closed subgroup of G . Denote by B_G and $B_H = B_G \cap H$ some Borel subgroups of G and H .*

If the pair (G, H) satisfies the pairing condition², and a variety X has a canonical B_G -splitting compatibly splitting a divisor Y , then X has a canonical B_H -splitting compatibly splitting Y .

In the paper of W. van der Kallen, the compatibility with Y is not explicitly stated, but the splitting of X is given by a morphism $\text{St}_G \otimes \text{St}_G \rightarrow H^0(X, \omega_X^{1-p})$ and if it splits a divisor Y , then it is given by a morphism $\text{St}_G \otimes \text{St}_G \rightarrow H^0(X, \omega_X^{1-p}((p-1)Y))$ (see [BrKu05, Theorem 1.4.10]). The pairing criterion gives a morphism $\text{St}_H \otimes \text{St}_H \rightarrow \text{St}_G \otimes \text{St}_G \rightarrow H^0(X, \omega_X^{1-p}((p-1)Y))$ and the compatibility follows by [BrKu05, Theorem 1.4.10] again.

Both in types A and D , this result is sufficient. Indeed, in type A , the pair is $(\text{SL}(\text{Im}N) \times \text{SL}(\text{Im}N), \text{SL}(\text{Im}N))$ and in type D , the pair is $(\text{SL}(\text{Im}N), \text{Sp}(\text{Im}N))$. The first pair satisfies the pairing criterion according to W. van der Kallen [vdK01, Example 8], as well as the second pair [vdK01, Remark 19]³.

We thus have a B -canonical splitting on $\tilde{\mathcal{F}}_{\mathbf{v}}$ compatible with all the divisors D_j and therefore with all the subvarieties $\tilde{\mathcal{F}}_{\mathbf{u}}$ (by [BrKu05, Proposition 1.2.1] and the fact that the varieties $\tilde{\mathcal{F}}_{\mathbf{u}}$ are intersections of such divisors). Applying [BrKu05, Theorem 4.1.17], we get a B -canonical splitting on $G \times^B \tilde{\mathcal{F}}_{\mathbf{v}} = \tilde{Y}$ compatible with all subvarieties $G \times^B \tilde{\mathcal{F}}_{\mathbf{u}} = \tilde{X}_{\mathbf{u}}$.

(iii) This is a direct application of Lemma 1.1.8 in [BrKu05] together with the fact that since Y is a Schubert variety, it is normal. \square

4.4 D -splitting

In this subsection, we prove that the previous splitting is a D -splitting with an explicit ample divisor D . For this we first need to compute the canonical divisor of the variety \tilde{Y} .

²This conditions says that, denoting by St_H and St_G the Steinberg modules of H and G , there is an H -module morphism $\text{St}_H \otimes \text{St}_H \rightarrow \text{St}_G \otimes \text{St}_G$ such that the composition with the evaluation $\text{St}_G \otimes \text{St}_G \rightarrow \mathbb{K}$ is nonzero. See [vdK01, Definition 6]. We will not really need this condition here.

³Remark that in type A the above results also follow from a more recent result (stated with compatibility of the splittings) proved by X. He and J.F. Thomsen [HeTh08, Theorem 7.2].

Let us first fix some notation. As we have seen, if $\mathbf{v} = (\alpha_{i_1}, \dots, \alpha_{i_k}, \dots, \alpha_{i_l})$ and if we denote by $\mathbf{v}[j]$ the subword $(\alpha_{i_1}, \dots, \alpha_{i_j})$ for $j \in [1, l]$, then the variety \tilde{Y} can be realized as a sequence of \mathbb{P}^1 -fibrations $\tilde{Y} = \tilde{X}_{\mathbf{v}} \rightarrow \tilde{X}_{\mathbf{v}[l-1]} \rightarrow \dots \rightarrow \tilde{X}_{\mathbf{v}[1]} \rightarrow G/B$. For all $j \in [1, l]$, there is a natural map $p_{X_{\mathbf{v}[j]}} \rightarrow \mathcal{F}$ and if $\mathcal{O}_{\mathcal{F}}(1)$ is the line bundle on \mathcal{F} defined by the Plücker embedding, we define $\mathcal{L}_{\mathbf{v}[j]} = p_{\tilde{X}_{\mathbf{v}[j]}}^*(\mathcal{O}_{\mathcal{F}}(1))$. We shall denote by $\mathcal{L}_{\tilde{Y}}$ and $\mathcal{L}_{\tilde{X}}$ the line bundles $\mathcal{L}_{\mathbf{v}}$ and $\mathcal{L}_{\mathbf{w}}$, respectively.

The following lemma is an easy modification of a well-known result on the canonical divisor of the Bott–Samelson resolution, see for example [BrKu05, Proposition 2.2.2] or [Kum02, Proposition 8.1.2]:

Lemma 4.8. *We have the equality $\omega_{\tilde{Y}}^{-1} = \mathcal{O}_{\tilde{Y}}(\partial\tilde{Y}) \otimes \mathcal{L}_{\tilde{Y}}$.*

Proof. We prove the following formula by induction over $j \in [0, l]$:

$$\omega_{\tilde{X}_{\mathbf{v}[j]}}^{-1} = \mathcal{O}_{\tilde{X}_{\mathbf{v}[j]}}(\partial\tilde{X}_{\mathbf{v}[j]}) \otimes \mathcal{L}_{\mathbf{v}[j]}.$$

For $j = 0$, we have $\tilde{X}_{\mathbf{v}[j]} = G/B$. The line bundle $\omega_{G/B}^{-1}$ is the line bundle $\mathcal{L}_{\mathbf{v}[j]}$ which is twice the ample line bundle defined by the Plücker embedding of G/B , since G/B is diagonally embedded into $\mathcal{F} = G'/B'$. Let us denote the fibration $\tilde{X}_{\mathbf{v}[j+1]} \rightarrow \tilde{X}_{\mathbf{v}[j]}$ by f and its section by σ . The induction follows from the formula:

$$\omega_{\tilde{X}_{\mathbf{v}[j+1]}} = f^* \omega_{\tilde{X}_{\mathbf{v}[j]}} \otimes \mathcal{O}_{\tilde{X}_{\mathbf{v}[j+1]}}(-\tilde{X}_{\mathbf{v}[j]}) \otimes \mathcal{L}_{\mathbf{v}[j+1]} \otimes f^* \sigma^* \mathcal{L}_{\mathbf{v}[j+1]}$$

which is a direct application of [Kum02, Lemma A-18], of the equality $\sigma^* \mathcal{L}_{\mathbf{v}[j+1]} = \mathcal{L}_{\mathbf{v}[j]}$ and the fact that for each j the divisor $\tilde{X}_{\mathbf{v}[j+1]}$ has the relative degree 1 for the fibration f . \square

Theorem 4.9. *There exists a \tilde{D} -splitting of \tilde{Y} compatibly splitting the subvarieties $\tilde{X}_{\mathbf{u}}$, where \mathbf{u} is a subword of \mathbf{v} and \tilde{D} is an effective divisor such that $\mathcal{O}_{\tilde{Y}}(\tilde{D}) = \mathcal{L}_{\tilde{Y}}^{\otimes p-1}$.*

Proof. Recall that in Theorem 4.5 we constructed a splitting φ of \tilde{Y} compatibly splitting the subvarieties $\tilde{X}_{\mathbf{u}}$ for a subword \mathbf{u} of \mathbf{v} . In particular, it is compatible with each of the divisors $\tilde{X}_{\mathbf{v}(j)}$ for $j \in [1, l]$, where $\mathbf{v}(j) = (\alpha_{i_1}, \dots, \widehat{\alpha_{i_j}}, \dots, \alpha_{i_l})$. By [BrKu05, Theorem 1.4.10], the splitting φ provides a $(p-1)\tilde{X}_{\mathbf{v}(j)}$ -splitting for all $j \in [1, l]$. We may thus write

$$\mathrm{div}(\varphi) = (p-1) \sum_{j=1}^l \tilde{X}_{\mathbf{v}(j)} + \tilde{D} = \partial\tilde{Y} + \tilde{D}$$

with $\mathcal{O}_{\tilde{Y}}(\tilde{D}) = \mathcal{L}_{\tilde{Y}}^{\otimes p-1}$ (compare with Lemma 4.8). But again by [BrKu05, Theorem 1.4.10], the splitting φ is a $\mathrm{div}(\varphi)$ -splitting. Now using [BrKu05, Remark 1.4.2 (u)] we get that φ is a \tilde{D} -splitting.

We next prove that the restriction of φ to $\tilde{X}_{\mathbf{v}(j)}$ is again a \tilde{D} -splitting for all $j \in [1, l]$. We thus consider $\varphi|_{\tilde{X}_{\mathbf{v}(j)}}$ the restriction of φ to $\tilde{X}_{\mathbf{v}(j)}$ given by the adjunction formula:

$$H^0(\tilde{Y}, \omega_{\tilde{Y}}^{1-p}((p-1)\tilde{X}_{\mathbf{v}(j)})) \rightarrow H^0(\tilde{X}_{\mathbf{v}(j)}, \omega_{\tilde{X}_{\mathbf{v}(j)}}).$$

We know that $\varphi|_{\tilde{X}_{\mathbf{v}(j)}}$ splits compatibly all the divisors $\tilde{X}_{\mathbf{v}(i)} \cap \tilde{X}_{\mathbf{v}(j)}$ of $\tilde{X}_{\mathbf{v}(j)}$ for $i \in [1, l]$ and $i \neq j$. In particular we get

$$\mathrm{div}(\varphi|_{\tilde{X}_{\mathbf{v}(j)}}) = (p-1) \sum_{i=1, i \neq j}^l \tilde{X}_{\mathbf{v}(i)}|_{\tilde{X}_{\mathbf{v}(j)}} + \tilde{D}'$$

with $\tilde{D}' = \tilde{D}|_{\tilde{X}_{\mathbf{v}(j)}}$ effective. By the above argument $\varphi|_{\tilde{X}_{\mathbf{v}(j)}}$ is a \tilde{D}' -splitting.

Finally, the result follows from the fact that for any subword \mathbf{u} of \mathbf{v} the variety $\tilde{X}_{\mathbf{u}}$ is the intersection of certain divisors $\tilde{X}_{\mathbf{v}(j)}$. \square

Corollary 4.10. *There exists a D -splitting of Y compatibly splitting all the subvarieties $X_{\mathbf{u}}$ for a subword \mathbf{u} of \mathbf{v} and an ample effective divisor D such that $\mathcal{O}_Y(D) = (\mathcal{O}_{\mathcal{F}}(1)|_Y)^{\otimes p-1}$.*

Proof. This is a direct application of [BrKu05, Lemma 1.4.5] to the map $p_{\tilde{Y}} : \tilde{Y} \rightarrow Y$. Indeed, in the previous Theorem, the divisor \tilde{D} is the pullback by $p_{\tilde{Y}}$ of a divisor D with the above property. We may apply Lemma 1.4.5 in [BrKu05] because Y is normal and the conclusion follows for the splitting of the varieties $X_{\mathbf{u}}$ because these varieties are the images of the varieties $\tilde{X}_{\mathbf{u}}$ under $p_{\tilde{Y}}$. \square

5 Normality

In this section we prove the results stated in the introduction. The proof will be similar to the proof of the same results for Schubert varieties as given in the book [BrKu05]. In order to pass from positive characteristic to characteristic zero, we shall use the results in [BrKu05, Section 1.6]. For this we need to realize the Springer fiber over \mathbb{Z} . This can be easily done by choosing a representative of the nilpotent element N in the normal Jordan form in its $\mathrm{GL}(\mathbb{K}^n)$ -orbit.

5.1 Some preliminary results

We prove the normality of all the varieties $X_{\mathbf{v}[j]}$ for $j \in [0, l]$ by induction over j . For this we need a more precise description of the geometry relating $\tilde{X}_{\mathbf{v}[j]}$ and $\tilde{X}_{\mathbf{v}[j+1]}$. Recall the construction of the variety $\tilde{X}_{\mathbf{v}[j+1]}$ from $\tilde{X}_{\mathbf{v}[j]}$ by the elementary construction 4.1 as the fiber product $\tilde{X}_{\mathbf{v}[j+1]} = \tilde{X}_{\mathbf{v}[j]} \times_{\mathcal{F}} \mathcal{F}_{\alpha_{i_{j+1}}}$. For a subvariety Z in \mathcal{F} we denote by $Z^{i_{j+1}}$ its image under the projection $p_{\alpha_{i_{j+1}}} : \mathcal{F} \rightarrow \mathcal{F}(\alpha_{i_{j+1}})$ (see Subsection 4.1). We have the equality $X_{\mathbf{v}[j+1]} = X_{\mathbf{v}[j]}^{i_{j+1}} \times_{\mathcal{F}} \mathcal{F}_{\alpha_{i_{j+1}}}$, so we obtain the following commutative diagram

$$\begin{array}{ccc}
 \tilde{X}_{\mathbf{v}[j+1]} = \tilde{X}_{\mathbf{v}[j]} \times_{\mathcal{F}} \mathcal{F}_{\alpha_{i_{j+1}}} & \xrightarrow{\tilde{a}} & \tilde{X}_{\mathbf{v}[j]} \\
 \downarrow b' & & \downarrow p_{\tilde{X}_{\mathbf{v}[j]}} \\
 X'_{\mathbf{v}[j+1]} := X_{\mathbf{v}[j]} \times_{\mathcal{F}} \mathcal{F}_{\alpha_{i_{j+1}}} & \xrightarrow{a'} & X_{\mathbf{v}[j]} \\
 \downarrow b & \swarrow & \downarrow p \\
 X_{\mathbf{v}[j+1]} = X_{\mathbf{v}[j]}^{i_{j+1}} \times_{\mathcal{F}} \mathcal{F}_{\alpha_{i_{j+1}}} & \xrightarrow{a} & X_{\mathbf{v}[j]}^{i_{j+1}} \\
 \downarrow p & \swarrow & \\
 X_{\mathbf{v}[j+1]}^{i_{j+1}} & &
 \end{array} \tag{2}$$

Lemma 5.1. *With the above notation,*

- (i) *the map $p_{\tilde{X}_{\mathbf{v}[j]}} : \tilde{X}_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}$ is birational for all $j \in [0, l]$;*
- (ii) *the map $p : X_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}^{i_{j+1}}$ is birational for all $j \in [0, l-1]$;*
- (iii) *we have the equality $X_{\mathbf{v}[j]}^{i_{j+1}} = X_{\mathbf{v}[j+1]}^{i_{j+1}}$ for all $j \in [0, l-1]$.*

Proof. (i) We prove this by descending induction on j . For $j = l$ the corresponding map is $p_{\tilde{Y}} : \tilde{Y} \rightarrow Y$, which is birational by Proposition 4.3.

Assume that $p_{\tilde{X}_{\mathbf{v}[j+1]}}$ is birational. This map is the composition of the top two left vertical arrows b and b' in the previous diagram. In particular these two maps b and b' are also birational. But the topmost right vertical arrow in the above diagram gives b' by fiber product. This map is $p_{\tilde{X}_{\mathbf{v}[j]}}$ and has to be birational.

(ii) By what we just proved the map b is birational. But it is a fiber product of the map $p : X_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}^{i_{j+1}}$, which has to be birational.

(iii) Recall that we have two maps $\varphi_{\alpha_{i_{j+1}}}$ and $\psi_{\alpha_{i_{j+1}}}$ from $\mathcal{F}_{\alpha_{i_{j+1}}}$ to \mathcal{F} obtained by forgetting one of the two subspaces corresponding to points in $\mathbb{G}(\alpha_{i_{j+1}})$ in $\mathcal{F}_{\alpha_{i_{j+1}}}$. The map a' corresponds to $\varphi_{\alpha_{i_{j+1}}}$ while b corresponds to $\psi_{\alpha_{i_{j+1}}}$. The composition of the two forgetful maps $\varphi_{\alpha_{i_{j+1}}}$ and $\psi_{\alpha_{i_{j+1}}}$ yield a map $\mathcal{F}_{\alpha_{i_{j+1}}} \rightarrow \mathcal{F}^{i_{j+1}}$. The maps $p \circ a'$ and $p \circ b$ correspond by fiber product to the maps $\psi_{\alpha_{i_{j+1}}} \circ \varphi_{\alpha_{i_{j+1}}}$ and $\psi_{\alpha_{i_{j+1}}} \circ \varphi_{\alpha_{i_{j+1}}}$, respectively. In particular these two maps are equal and the result follows. \square

5.2 Proof of Theorem 1.1

We prove by induction over j that $X_{\mathbf{v}[j]}$ is normal. For $j = 0$, we have $X_{\mathbf{v}[0]} \simeq G/B$, which is normal. Let $j > 0$ and assume that $X_{\mathbf{v}[j]}$ is normal. The map $a : X_{\mathbf{v}[j+1]} \rightarrow X_{\mathbf{v}[j]}^{i_{j+1}}$ is a \mathbb{P}^1 -fibration. Thus to prove the normality of $X_{\mathbf{v}[j+1]}$ we only need to prove the normality of $X_{\mathbf{v}[j]}^{i_{j+1}}$. But the map $p : X_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}^{i_{j+1}}$ is birational and surjective (Lemma 5.1 (ii)), and $X_{\mathbf{v}[j]}$ is normal by the induction hypothesis, so we only need to prove the equality

$$p_* \mathcal{O}_{X_{\mathbf{v}[j]}} = \mathcal{O}_{X_{\mathbf{v}[j]}^{i_{j+1}}}.$$

This will be done using the following lemma (see [BrKu05, Lemma 3.3.3]):

Lemma 5.2. *Let $f : X \rightarrow Y$ be a surjective morphism between projective schemes, and let \mathcal{L} be an ample line bundle on Y . Assume that the map $H^0(Y, \mathcal{L}^\nu) \rightarrow H^0(X, f^* \mathcal{L}^\nu)$ is an isomorphism for ν large enough. Then $f_* \mathcal{O}_X = \mathcal{O}_Y$.*

Consider \mathcal{L} ample on $X_{\mathbf{v}[j]}^{i_{j+1}}$ and the following commutative diagram

$$\begin{array}{ccc} H^0(X_{\mathbf{v}[j+1]}^{i_{j+1}}, \mathcal{L}) & \longrightarrow & H^0(X_{\mathbf{v}[j+1]}, p^* \mathcal{L}) \\ \parallel & & \downarrow \\ H^0(X_{\mathbf{v}[j]}^{i_{j+1}}, \mathcal{L}) & \longrightarrow & H^0(X_{\mathbf{v}[j]}, p^* \mathcal{L}). \end{array}$$

But since $X_{\mathbf{v}[j+1]}$ is D -split compatibly with $X_{\mathbf{v}[j]}$ and D is ample, we get by [BrKu05, Theorem 1.4.8] that the right vertical map is surjective. Moreover, the map $p : X_{\mathbf{v}[j+1]} \rightarrow X_{\mathbf{v}[j]}^{i_{j+1}}$ is a \mathbb{P}^1 -fibration, so the top horizontal map is also surjective. We obtain that the lower horizontal map is surjective. It is injective since the map $p : X_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}^{i_{j+1}}$ is surjective. We may thus apply the previous lemma and deduce the normality.

Remark 5.3. This proof works for \mathbb{K} of positive characteristic, but relies only on vanishing of cohomology and surjectivity of restrictions on cohomology, which pass, by semi-continuity, to characteristic zero. The same proof therefore works for $\text{Char}(\mathbb{K}) = 0$.

5.3 Proof of Theorem 1.2

Recall the definition of a rational morphism and of rational singularities for $\text{Char}(\mathbb{K}) = 0$:

Definition 5.4. (1) A morphism of schemes $f: X \rightarrow Y$ is called *rational* if $f_*\mathcal{O}_X = \mathcal{O}_Y$ and all its higher direct images vanish: $R^i f_*\mathcal{O}_X = 0$ for $i > 0$.

(11) Assume $\text{Char}(\mathbb{K}) = 0$. A normal variety X has *rational singularities* if there exists a rational birational proper morphism $\pi: \tilde{X} \rightarrow X$.

We first prove the following

Lemma 5.5. *For all $j \in [0, l]$, the map $p_{\tilde{X}_{\mathbf{v}[j]}}: \tilde{X}_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}$ is a rational morphism.*

Proof. We prove this lemma by induction over j . For $j = 0$, we have $p_{\tilde{X}_{\mathbf{v}[0]}}: \tilde{X}_{\mathbf{v}[0]} \rightarrow X_{\mathbf{v}[0]}$ is an isomorphism. Let $j > 0$ and assume that $p_{X_{\mathbf{v}[j]}}$ is rational. Then, since b' is obtained by fiber product from $p_{\tilde{X}_{\mathbf{v}[j]}}$, we see that b' is rational. So, to prove the rationality of $p_{\tilde{X}_{\mathbf{v}[j+1]}}$ we only need to prove the rationality of b . Since b is obtained by fiber product from p , we only need to prove the rationality of $p: X_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}^{i_{j+1}}$. But p is birational and $X_{\mathbf{v}[j]}^{i_{j+1}}$ is normal (see the proof of the normality of X), therefore by the Zariski Main Theorem we obtain the equality $p_*\mathcal{O}_{X_{\mathbf{v}[j]}} = \mathcal{O}_{X_{\mathbf{v}[j]}^{i_{j+1}}}$. We need to prove the vanishing of the higher direct images. For this we embed $X_{\mathbf{v}[j]}$ in \mathcal{F} and $X_{\mathbf{v}[j]}^{i_{j+1}}$ in $\mathcal{F}(\alpha_{i_{j+1}})$. We have a commutative diagram

$$\begin{array}{ccc} X_{\mathbf{v}[j]} & \hookrightarrow & \mathcal{F} \\ \downarrow p & & \downarrow \\ X_{\mathbf{v}[j]}^{i_{j+1}} & \hookrightarrow & \mathcal{F}(\alpha_{i_{j+1}}) \end{array}$$

and in particular the fibers of both morphisms are at most one-dimensional, thus $R^i p_*\mathcal{O}_{X_{\mathbf{v}[j]}} = 0$ for $i \geq 2$. Now the surjection $\mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{O}_{X_{\mathbf{v}[j]}}$ induces a surjection $R^1 p_*\mathcal{O}_{\mathcal{F}} \rightarrow R^1 p_*\mathcal{O}_{X_{\mathbf{v}[j]}}$. But since the second vertical map is a \mathbb{P}^1 -fibration, we have $R^1 p_*\mathcal{O}_{\mathcal{F}} = 0$ and the result follows. \square

Let us now prove the vanishing $R^i p_{\tilde{X}_{\mathbf{v}[j]}*} \omega_{\tilde{X}_{\mathbf{v}[j]}} = 0$ for $i > 0$. For this we use the following direct application of Theorem 1.2.12 from [BrKu05]:

Lemma 5.6. *Let $f: X \rightarrow Y$ is a proper birational morphism with X smooth. Assume that φ is a splitting for X compatibly splitting a divisor Z such that the exceptional locus of f is set-theoretically contained in Z . Then we have $R^i f_*\mathcal{O}_X(-Z) = 0$ for $i > 0$.*

We want to apply this lemma to the map $p: \tilde{X}_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}$, the splitting constructed above, and the divisor $Z = \partial\tilde{X}_{\mathbf{v}[j]}$. For this we only need to check that the exceptional locus of $p_{\tilde{X}_{\mathbf{v}[j]}}$ is contained in $\partial\tilde{X}_{\mathbf{v}[j]}$. But the map $\tilde{X}_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}$ decomposes as follows:

$$\tilde{X}_{\mathbf{v}[j]} = G \times^B \tilde{\mathcal{F}}_{\mathbf{v}[j]} \rightarrow G \times^B \mathcal{F}_{\mathbf{v}[j]} \rightarrow X_{\mathbf{v}[j]}$$

and this map is G -equivariant. Furthermore, the complement to $\partial\tilde{\mathcal{F}}_{\mathbf{v}[j]}$ in $\tilde{\mathcal{F}}_{\mathbf{v}[j]}$ is a B -equivariant dense open subset thus the complement of $\partial\tilde{X}_{\mathbf{v}[j]}$ in $\tilde{X}_{\mathbf{v}[j]}$ is a G -equivariant dense open subset. The map is therefore an isomorphism on this open subset, and the exceptional locus is contained in $\partial\tilde{X}_{\mathbf{v}[j]}$. By the previous lemma, we get the vanishing

$$R^i p_{\tilde{X}_{\mathbf{v}[j]}*} \mathcal{O}_{\tilde{X}_{\mathbf{v}[j]}}(-\partial\tilde{X}_{\mathbf{v}[j]}) = 0 \text{ for } i > 0.$$

But from Lemma 4.8, we have $\omega_{\tilde{X}_{\mathfrak{v}[j]}} = \mathcal{O}_{\tilde{X}_{\mathfrak{v}[j]}}(-\partial\tilde{X}_{\mathfrak{v}[j]}) \otimes p_{\tilde{X}_{\mathfrak{v}[j]}}^*(\mathcal{O}_{\mathcal{F}}(1)|_{X_{\mathfrak{v}[j]}})$. Thus by projection formula, we get:

$$R^i p_{\tilde{X}_{\mathfrak{v}[j]*}} \omega_{\tilde{X}_{\mathfrak{v}[j]}} = R^i p_{\tilde{X}_{\mathfrak{v}[j]*}} \mathcal{O}_{\tilde{X}_{\mathfrak{v}[j]}}(-\partial\tilde{X}_{\mathfrak{v}[j]}) \otimes (\mathcal{O}_{\mathcal{F}}(1)|_{X_{\mathfrak{v}[j]}}) = 0 \text{ for } i > 0.$$

This completes the proof of Theorem 1.2. Corollary 1.3 follows from general results on rational resolutions, see [BrKu05, Lemma 3.4.2], and Corollary 1.4 follows from the definition of rational singularities and Lemma 5.5.

Remark 5.7. The proof of Lemma 5.5 works for any characteristic (once the normality is proved). For $\text{Char}(\mathbb{K}) = 0$ we do not need to prove the above vanishing $R^i p_{\tilde{X}_{\mathfrak{v}[j]*}} \omega_{\tilde{X}_{\mathfrak{v}[j]}} = 0$ for $i > 0$. This result follows automatically from Grauert–Riemenschneider Theorem [GrRi70].

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